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Winkler 地基上材料非线性矩形薄板 主参数共振-主共振研究

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摘 要: 研究 Winkler 地基上材料非线性矩形薄板的参数共振-主共振问题。建立 Winkler 地基上材料非线性矩形薄板的动力学方程。利用 Galerkin 方法将其转化为非线性振动方程。应用非线性振动的多尺度法求得系统满足主参数共振-主共振条件的近似解, 并进行数值计算。分析激励、调谐值、阻尼系数、非线性参数、几何参数对共振响应曲线的影响。

关 键 词: 非线性; Galerkin 方法; 多尺度法; 主参数共振-主共振; Winkler 地基

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Study on primary parametric resonance with primary resonance of thin rectangular plate on Winkler foundation considering material nonlinearity

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Abstract: In order to study on primary parametric resonance with primary resonance of thin rectangular plate on Winkler foundation subjected to harmonic excitation considering material nonlinearity, its nonlinear dynamical equation is established and transformed to nonlinear vibration equation based on Galerkin's method. By using the method of multiple scales for nonlinear vibration, the approximate solution of primary parametric resonance and primary resonance of the system is acquired. Numerical computation is carried out to analyze the influences of excitation, detuning, damping, nonlinear parameters and geometric parameters on the system.

Key words: nonlinearity; Galerkin's method; method of multiple scales; primary parametric resonance and primary resonance; Winkler foundation

1 引 言

在公路路面、机场跑道、停机场、工业地坪及建筑基础等多种工程中都会遇到地基板的计算分析, 所以弹性地基板的研究具有重要意义, 同时也引起很多学者的重视。文[1]对线性地基上四边自由矩形薄板振动问题进行了研究, 给出了静动态挠度的计算公式。文[2-4]研究弹性地基上圆板的非线性振动问题。文[5]研究非线性弹性地基上圆板分岔问题及混沌运动的条件。文[6、7]研究非线性弹性地基上矩形板的复杂混沌运动和参数共振问题。文[8]给出材料非线性矩形板的本构方程, 并研

究了自由振动问题。最近, 科学家将纤维加入混凝土中制成一种重量轻、抗裂缝、耐用持久的可弯曲材料, 此种材料的应力-应变之间存在非线性关系。这种新型材料被称作“工程水泥复合材料(ECC)”。以下研究置于 Winkler 地基上四边简支受均布简谐激励材料非线性矩形薄板的主参数共振-主共振问题。

2 基本方程

考虑图 1 所示的四边简支工程水泥复合材料矩形薄板, 板厚为 h , 长、宽分别为 a 和 b , 在其四周边界上均匀分布简谐压力 N_1 和 N_2 , 在横向受有

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简谐激励 $q \cos \Omega_2 t$, 考虑 Winkler 地基反力 $k_1 w$ 和 阻尼力。

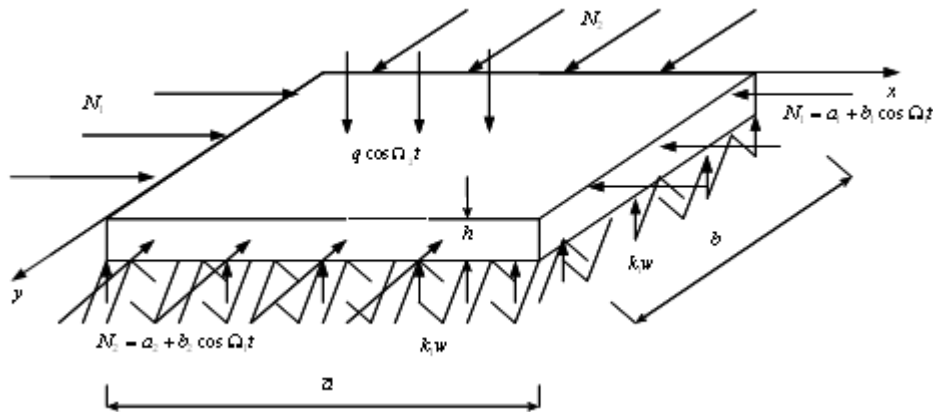


图 1 力学模型

Fig.1 Mechanical model

板单元的动力方程为

$$\left. \begin{aligned} \frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} &= 0, \quad \frac{\partial N_{yx}}{\partial x} + \frac{\partial N_y}{\partial y} = 0; \\ \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x &= 0, \quad \frac{\partial M_{yx}}{\partial x} + \frac{\partial M_y}{\partial y} - Q_y = 0; \\ \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + N_x \frac{\partial^2 W}{\partial x^2} + N_y \frac{\partial^2 W}{\partial y^2} + 2N_{xy} \frac{\partial^2 W}{\partial x \partial y} - \\ \rho h \frac{\partial^2 W}{\partial t^2} - k_1 W - c \frac{\partial W}{\partial t} + q \cos \Omega_2 t &= 0 \end{aligned} \right\} \quad (1)$$

式中: N_x 、 N_y 、 N_{xy} 为板单元单位长度上的薄膜力; M_x 、 M_y 、 M_{xy} 为板单元单位长度上的内力矩; Q_x 、 Q_y 为板单元单位长度上垂直板平面的剪力; ρ 为材料的密度; W 为横向挠度; c 为阻尼系数; k_1 为地基系数, 且有

$$\left. \begin{aligned} N_x &= \int_{-h/2}^{h/2} \sigma_x dz; \quad N_y = \int_{-h/2}^{h/2} \sigma_y dz; \\ N_{xy} &= \int_{-h/2}^{h/2} \tau_{xy} dz; \quad M_x = \int_{-h/2}^{h/2} \sigma_x z dz; \\ M_y &= \int_{-h/2}^{h/2} \sigma_y z dz; \quad M_{xy} = \int_{-h/2}^{h/2} \tau_{xy} z dz \end{aligned} \right\} \quad (2)$$

板单元动力方程的前 2 式是独立的, 可以认为

$$N_x = -N_1; \quad N_y = -N_2; \quad N_{xy} = 0 \quad (3)$$

将式(3)代入式(1)中的第 5 式可得:

$$\left. \begin{aligned} \frac{\partial^2 M_x}{\partial x^2} + 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} - N_1 \frac{\partial^2 W}{\partial x^2} - N_2 \frac{\partial^2 W}{\partial y^2} - \\ \rho h \frac{\partial^2 W}{\partial t^2} - k_1 W - c \frac{\partial W}{\partial t} + q \cos \Omega_2 t &= 0 \end{aligned} \right\} \quad (4)$$

对单向应力状态, 设材料应力 $\bar{\sigma}$ -应变 $\bar{\varepsilon}$ 的非

线性方程如式 (5) (许多非线性弹性材料具有类似的本构关系):

$$\bar{\sigma} = E(\bar{\varepsilon} - B\bar{\varepsilon}^3) \quad (5)$$

式中: E 为材料的初始弹性模量; B 为另一个新的材料常数, 且 B 可为正值也可为负值, 根据弹性力学理论, 考虑材料非线性薄板的物理方程和几何方程, 得系统的动力学方程:

$$\begin{aligned} D \nabla^4 W - 3 \bar{B} D \left\{ \left(\frac{\partial^2 W}{\partial x^2} \right)^2 \left(\frac{\partial^4 W}{\partial x^4} \right) + \left(\frac{\partial^2 W}{\partial y^2} \right)^2 \left(\frac{\partial^4 W}{\partial y^4} \right) + \right. \\ \left. 2 \left(\frac{\partial^2 W}{\partial x^2} \right) \left(\frac{\partial^3 W}{\partial x^3} \right)^2 + 2 \left(\frac{\partial^2 W}{\partial y^2} \right) \left(\frac{\partial^3 W}{\partial y^3} \right)^2 + \right. \\ \left. \mu \left[\Delta^2 W \left(\frac{\partial^4 W}{\partial x^2 \partial y^2} \right) + 2 \left(\frac{\partial^2 W}{\partial x^2} \right) \left(\frac{\partial^3 W}{\partial x^2 \partial y} \right)^2 + \right. \right. \\ \left. \left. 2 \left(\frac{\partial^2 W}{\partial y^2} \right) \left(\frac{\partial^3 W}{\partial x \partial y^2} \right)^2 + 4(1-\mu) \left(\frac{\partial^2 W}{\partial x \partial y} \right)^2 \cdot \right. \right. \\ \left. \left. \left(\frac{\partial^3 W}{\partial x^2 \partial y} \right) \left(\frac{\partial^3 W}{\partial x \partial y^2} \right) + 2(1-\mu) \left(\frac{\partial^2 W}{\partial x \partial y} \right)^2 \left(\frac{\partial^4 W}{\partial x^2 \partial y^2} \right) \right] + \right. \\ \left. N_1 \frac{\partial^2 W}{\partial x^2} + N_2 \frac{\partial^2 W}{\partial y^2} + \rho h \frac{\partial^2 W}{\partial t^2} + \right. \\ \left. k_1 W + c \frac{\partial W}{\partial t} = q \cos \Omega_2 t \right\} \quad (6) \end{aligned}$$

其中 D 为板的抗弯刚度,

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}; \quad \Delta^2 = \left(\frac{\partial^2}{\partial x^2} \right)^2 + \left(\frac{\partial^2}{\partial y^2} \right)^2;$$

$$\bar{B} = \frac{3h^2}{20}B; \quad D = \frac{Eh^3}{12(1-\mu^2)}.$$

四边简支板的边界条件为

$$\left. \begin{aligned} x=0, \quad \bar{a}W = \frac{\partial^2 W}{\partial x^2} = 0; \\ y=0, \quad \bar{a}W = \frac{\partial^2 W}{\partial y^2} = 0 \end{aligned} \right\} \quad (7)$$

取位移模式形如

$$W(x, y, t) = U(t) \sin \frac{\pi x}{\bar{a}} \sin \frac{\pi y}{b} \quad (8)$$

显然, 由式(8)给的位移模式满足四边简支的边界条件, 由 Galerkin 原理得系统的非线性振动方程:

$$\ddot{U} + 2\alpha'\dot{U} + \omega_0^2 U - g'U \cos \Omega_1 t - k'U^3 = F \cos \Omega_2 t \quad (9)$$

其中

$$\left. \begin{aligned} \alpha' &= \frac{c}{2\rho h}; \\ D \left[\left(\frac{\pi}{\bar{a}} \right)^2 + \left(\frac{\pi}{b} \right)^2 \right]^2 - a_1 - a_2 + k_1 \\ \omega_0^2 &= \frac{\rho h}{\rho h}; \\ g' &= \left(\frac{b_1}{\bar{a}^2} + \frac{b_2}{b^2} \right) \frac{\pi^2}{\rho h}; \\ k' &= \frac{9\bar{B}D}{16\rho h} \left\{ \left(\frac{\pi}{\bar{a}} \right)^8 + \left(\frac{\pi}{b} \right)^8 + \mu \left(\frac{\pi}{\bar{a}} \right)^2 \left(\frac{\pi}{b} \right)^2 \cdot \right. \\ &\quad \left. \left[\left(\frac{\pi}{\bar{a}} \right)^4 + \left(\frac{\pi}{b} \right)^4 \right] + 2(1-\mu) \left(\frac{\pi}{\bar{a}} \right)^4 \left(\frac{\pi}{b} \right)^4 \right\} \end{aligned} \right\} \quad (10)$$

3 主参数共振-主共振的稳态解方程

因为研究主参数共振-主共振问题, 在阻尼项、非线性项、参数激励项及强迫激励项前冠以小参数 ε , 由式(9)得:

$$\ddot{U} + 2\varepsilon\alpha'\dot{U} + \omega_0^2 U - \varepsilon g'U \cos \Omega_1 t - \varepsilon k'U^3 = \varepsilon F \cos \Omega_2 t \quad (11)$$

令 $\Omega_1 t = 2T$, $\Omega_2 t = T$, 则

$$\left. \begin{aligned} \frac{dU}{dt} &= \frac{dU}{dT} \frac{dT}{dt} = \Omega_2 \frac{dU}{dT}; \\ \frac{d^2 U}{dt^2} &= \Omega_2^2 \frac{d^2 U}{dT^2} \end{aligned} \right\} \quad (12)$$

把式(12)代入式(11)得

$$\ddot{U} + \omega^2 U = \varepsilon (kU^3 - 2\alpha'\dot{U} + gU \cos 2T + f \cos T) \quad (13)$$

其中

$$\left. \begin{aligned} k &= \frac{k'}{\Omega_2^2}, \quad g = \frac{g'}{\Omega_2^2}; \\ \omega^2 &= \frac{\omega_0^2}{\Omega_2^2}; \quad \alpha = \frac{\alpha'}{\Omega_2}; \quad f = \frac{F}{\Omega_2^2} \end{aligned} \right\} \quad (14)$$

为了简化研究且不失一般性设 $\omega = 1 + \varepsilon\sigma$, 这里 σ 是调谐值, 采用多尺度法选用两个时间尺度 (T_0 , T_1), 则有 $\cos T = \cos 2T_0$, $\cos T = \cos T_0$, 设满足主参数共振-主共振的一次近似解为

$$U(T) = U_0(T_0, T_1) + \varepsilon U_1(T_0, T_1) \quad (15)$$

将式(15)代入式(13)并考虑 $\omega = 1 + \varepsilon\sigma$ 比较 ε 同次幂的系数, 由多尺度法得到确定主参数共振-主共振关于振幅 a 和相位 φ 的一次近似解微分方程:

$$\left. \begin{aligned} D_1 a &= -\alpha a - \frac{g}{4} a \sin 2\varphi - \frac{f}{2} \sin \varphi; \\ a D_1 \varphi &= \sigma a - \frac{3}{8} k a^3 - \frac{g}{4} a \cos 2\varphi - \frac{f}{2} \cos \varphi \end{aligned} \right\} \quad (16)$$

在上两式中令 $D_1 a = D_1 \varphi = 0$, 得到的主参数共振-主共振稳态解是超越代数方程组, 由此仍无法得到振幅 a 和相位 φ 的解析表达式, 同时数学计算也存在数学上的收敛性和多值性的困难, 为此对方程组进行变换。设

$$w = a \cos \varphi; \quad v = a \sin \varphi \quad (17)$$

则有

$$\begin{aligned} D_1 w &= \cos \varphi D_1 a - a \sin \varphi D_1 \varphi = \\ &= -\left(\alpha a + \frac{ag}{4} \sin 2\varphi + \frac{f}{2} \sin \varphi \right) \cos \varphi - \sin \varphi \cdot \\ &\quad \left(\sigma a - \frac{3}{8} k a^3 - \frac{ag}{4} \cos 2\varphi - \frac{f}{2} \cos \varphi \right) = \\ &= -\alpha a \cos \varphi - \sigma a \sin \varphi + \frac{3}{8} k a^3 \sin \varphi + \\ &\quad \frac{ag}{4} \cos 2\varphi \sin \varphi - \frac{ag}{4} \sin 2\varphi \cos \varphi = \\ &= -\alpha a \cos \varphi - \sigma a \sin \varphi + \frac{3}{8} k a^3 \sin \varphi - \frac{ag}{4} \sin \varphi = \\ &= -\alpha w - \sigma v + \frac{3}{8} k a^2 v - \frac{g}{4} v = \\ &= -\alpha w - v \left(\sigma - \frac{3}{8} k a^2 + \frac{g}{4} \right) \end{aligned} \quad (18)$$

$$\begin{aligned}
D_1 v &= \sin \varphi D_1 a + a \cos \varphi D_1 \varphi = \\
& - \left(\alpha a + \frac{ag}{4} \sin 2\varphi + \frac{f}{2} \sin \varphi \right) \sin \varphi + \cos \varphi \cdot \\
& \left(\sigma a - \frac{3}{8} ka^3 - \frac{ag}{4} \cos 2\varphi - \frac{f}{2} \cos \varphi \right) = \\
& - \alpha a \sin \varphi + \sigma a \cos \varphi - \frac{3}{8} ka^3 \cos \varphi - \\
& \frac{ag}{4} \cos \varphi - \frac{f}{2} = \\
& - \alpha v + \sigma w - \frac{3}{8} ka^2 w - \frac{g}{4} w - \frac{f}{2} = \\
& - \alpha v + w \left(\sigma - \frac{3}{8} ka^2 - \frac{g}{4} \right) - \frac{f}{2}
\end{aligned} \quad (19)$$

式中 $w^2 + v^2 = a^2$, 令 $D_1 w = 0$, $D_1 v = 0$ 得主参共振-主共振稳态解对应的方程组

$$\left. \begin{aligned}
\alpha w + v \left(\sigma - \frac{3}{8} ka^2 + \frac{g}{4} \right) &= 0; \\
\alpha v - w \left(\sigma - \frac{3}{8} ka^2 - \frac{g}{4} \right) + \frac{f}{2} &= 0
\end{aligned} \right\} \quad (20)$$

从上式中解出 w 和 v 得

$$\left. \begin{aligned}
w &= \frac{-\frac{f}{2} \left(\sigma - \frac{3}{8} ka^2 + \frac{g}{4} \right)}{\left(\frac{g}{4} \right)^2 - \left(\sigma - \frac{3}{8} ka^2 \right)^2 - \alpha^2} \\
v &= \frac{\frac{f}{2} \alpha}{\left(\frac{g}{4} \right)^2 - \left(\sigma - \frac{3}{8} ka^2 \right)^2 - \alpha^2}
\end{aligned} \right\} \quad (21)$$

由 $w^2 + v^2 = a^2$ 得

$$\left[\frac{-\frac{f}{2} \left(\sigma - \frac{3}{8} ka^2 + \frac{g}{4} \right)}{\left(\frac{g}{4} \right)^2 - \left(\sigma - \frac{3}{8} ka^2 \right)^2 - \alpha^2} \right]^2 + \left[\frac{\frac{f}{2} \alpha}{\left(\frac{g}{4} \right)^2 - \left(\sigma - \frac{3}{8} ka^2 \right)^2 - \alpha^2} \right]^2 = a^2$$

整理得

$$J_0 a^{10} + J_1 a^8 + J_2 a^6 + J_3 a^4 + J_4 a^2 + J_5 = 0 \quad (22)$$

其中 $J_0 = -\frac{81}{4096} k^4$; $J_1 = \frac{27}{128} \sigma k^3$ 。

$$\begin{aligned}
J_2 &= \frac{9}{512} g^2 k^2 - \frac{9}{32} \alpha^2 k^2 - \frac{27}{32} \sigma^2 k^2; \\
J_3 &= \frac{9}{256} f^2 k^2 - \frac{3}{32} g^2 \sigma k + \frac{3}{2} \sigma^3 k + \frac{3}{2} \alpha^2 \sigma k; \\
J_4 &= -\frac{3}{64} f^2 g k - \frac{3}{16} f^2 \sigma k + \frac{1}{8} g^2 \sigma^2 + \\
& \frac{1}{8} g^2 \alpha^2 - 2\alpha^2 \sigma^2 - \frac{1}{256} g^4 - \alpha^4 - \sigma^4; \\
J_5 &= \frac{1}{64} f^2 g^2 + \frac{1}{8} f^2 g \sigma + \frac{1}{4} f^2 \sigma^2 + \frac{1}{4} f^2 \alpha^2
\end{aligned} \quad (23)$$

4 数值计算与结果分析

以下计算中 Winkler 地基上矩形板的取值为:

$\bar{a} = b = 5 \text{ m}$, $h = 0.25 \text{ m}$, $a_1 = a_2 = 600 \text{ N}$, $c = 0.1$, $E = 20 \text{ GPa}$, $q = 0.3 \text{ N}$, $\mu = 0.15$, $\rho = 2.5 \text{ g/cm}^3$, $B = 2\,000$, $b_1 = b_2 = 0.5 \text{ kN}$, $k_1 = 4 \times 10^8 \text{ N/m}$ 。

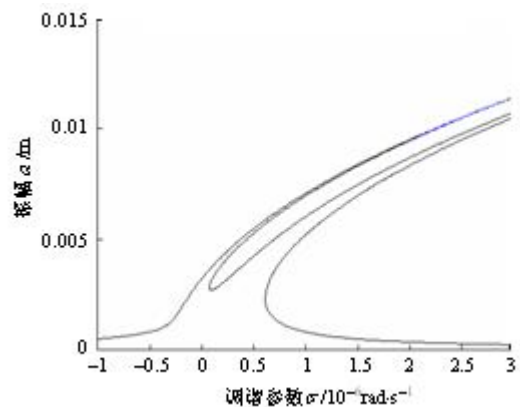


图2 幅频响应曲线, $c = 0.1$ $b_1 = b_2 = 500 \text{ N}$
Fig.2 Amplitude frequency response curves,
 $c = 0.1$ $b_1 = b_2 = 500 \text{ N}$

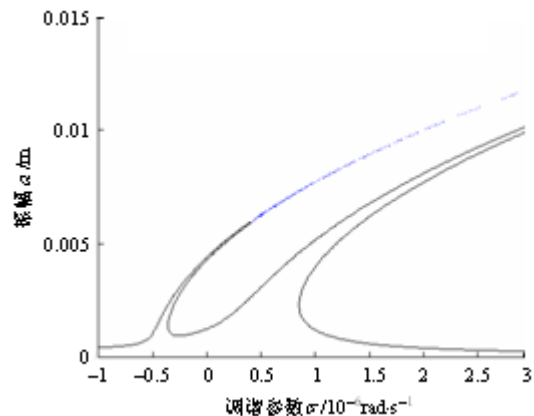


图3 幅频响应曲线, $b_1 = b_2 = 1\,000 \text{ N}$
Fig.3 Amplitude frequency response curves,
 $b_1 = b_2 = 1\,000 \text{ N}$

按式 (22) 应用 Matlab 语言可以计算系统主参

数共振-主共振的各种响应曲线,分析系统参数对响应曲线的影响。

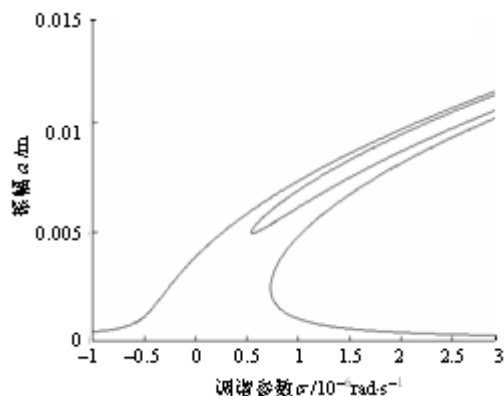


图4 幅频响应曲线, $c = 0.4$ $b_1 = b_2 = 1\ 000\ \text{N}$
Fig.4 Amplitude frequency response curves,
 $c = 0.4$ $b_1 = b_2 = 1\ 000\ \text{N}$

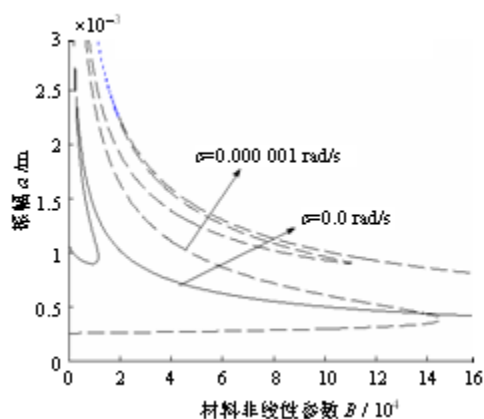


图5 振幅-B 响应曲线
Fig.5 Amplitude B response curves

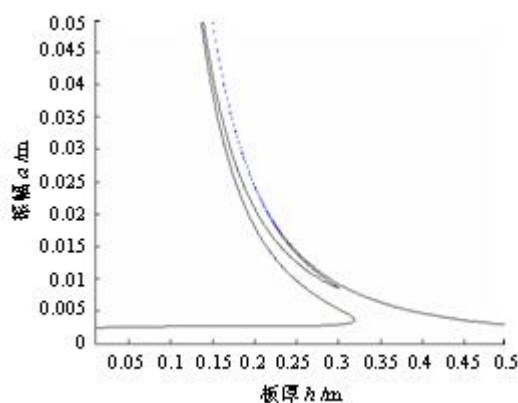


图6 振幅板厚响应曲线, $\sigma = 0.000\ 001$
Fig.6 Thickness-amplitude response curves,
 $\sigma = 0.000\ 001$

图2~4 为3组参数下的幅频响应曲线。由以上3图可知,主参数共振-主共振曲线存在跳跃现象和滞后现象。由图2和图3可知,增大参数激励幅值

可使“双峰”增强,由图3和图4可知,增大阻尼可是“双峰”减弱。图5当改变系统的调谐值时,考虑振幅与材料非线性参数 B 之间的关系。由图可见,响应曲线存在跳跃现象和滞后现象,并且随着调谐值的减小振动幅值减小。图6为调谐值取0.000 001时振幅与板厚度之间的关系,曲线具有跳跃现象和滞后现象。

图7为调谐值取零时振幅与板的几何尺寸之间的关系,曲线具有跳跃现象和滞后现象。图8为调谐值取不同值时振幅与参数激励幅值的响应曲线,具有跳跃现象和滞后现象。图9为改变材料非线性参数 B 时,考虑阻尼与振幅之间的关系,曲线具有跳跃现象和滞后现象,且随着 B 的增加共振幅值减小。

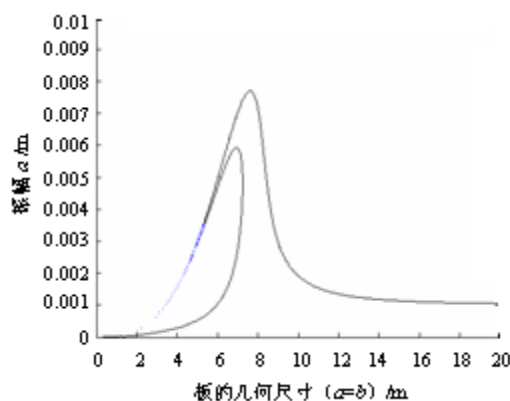


图7 振幅板的几何尺寸响应曲线, $\sigma = 0$, $q = 0.05\ \text{N}$
Fig.7 Geometric parameter amplitude response curve
 $\sigma = 0$, $q = 0.05\ \text{N}$

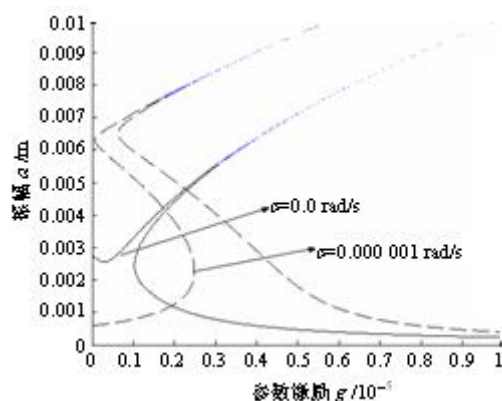


图8 振幅参数激励幅值响应曲线
Fig.8 Amplitude parameter excitation curves

5 结 论

(1) 主参数共振-主共振幅频响应曲线存在双峰值,有2次跳跃现象和滞后现象。(2) 随着参数

激励幅值的改变,主参数共振-主共振力幅响应曲线均具有跳跃和滞后现象。(3)系统参数对主参数共振-主共振响应曲线影响显著。

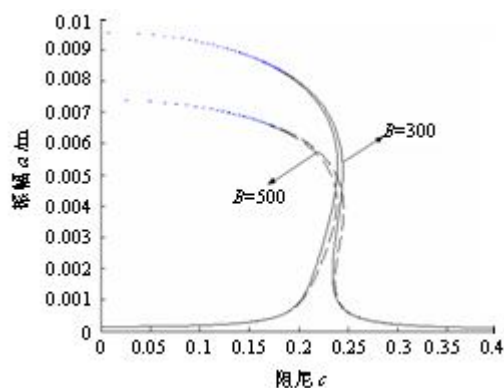


图9 振幅阻尼响应曲线, $\sigma=10^{-7}$, $q=0.01$ N

Fig.9 Amplitude damping response curves, response curve $\sigma=10^{-7}$, $q=0.01$ N

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